

Young's modulus, shear modulus, bulk modulus and Poisson's ratio

Definition and measurement. Figure 1 shows a typical tensile stress-strain curve. The initial part, up to the yield strength σ_y or elastic limit σ_{el} , defined under *Yield strength (elastic limit)*, is linear (Hooke's law), and it is elastic, meaning that the strain is recoverable – the material returns to its original shape when the stress is removed. Stresses above the elastic limit cause permanent deformation or fracture (see notes for **Yield strength (elastic limit)** and **Fracture toughness**).

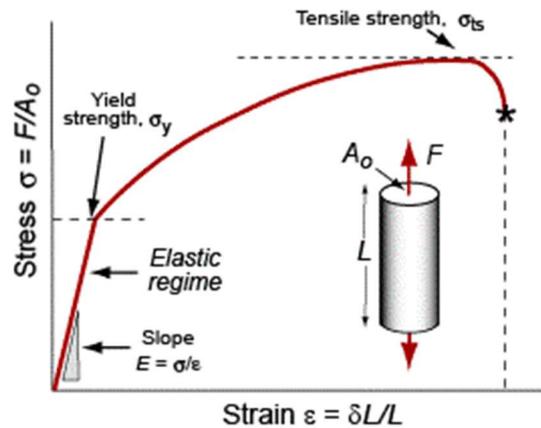


Figure 1. A tensile stress-strain curve.

Within the linear elastic regime, strain is proportional to stress, but stress can be applied in more than one way (Figure 2). The tensile stress σ produces a proportional tensile strain ϵ :

$$\sigma = E \epsilon$$

and the same is true in compression. The constant of proportionality, E , is called Young's modulus. Similarly, a shear stress σ_s causes a proportional shear strain γ

$$\sigma_s = G \gamma$$

and a pressure p results in a proportional fractional volume change (or "dilatation") Δ :

$$p = K \Delta$$

where G is the shear modulus and K the bulk modulus. All three of these moduli have the same dimensions as stress, that of force per unit area (N/m^2 or Pa). It is convenient to use a larger unit, that of 10^9 Pa, Giga-Pascals, or GPa.

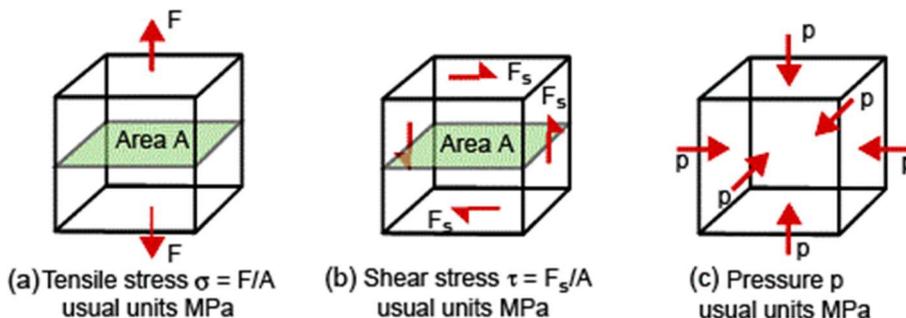


Figure 2. (a) Tensile stress. (b) Shear stress. (c) Hydrostatic pressure.

Young's modulus, the shear modulus, and the bulk modulus are related, but to relate them we need one more quantity, Poisson's ratio. When stretched in one direction, a material generally

contracts in the other two directions. Poisson's ratio, ν , is the negative of the ratio of the lateral or transverse strain, ϵ_{tr} , to the axial strain, ϵ , in tensile loading:

$$\nu = -\frac{\epsilon_{tr}}{\epsilon}$$

You might think that the way to measure the elastic modulus of a material would be to apply a small stress (to be sure to remain in the linear-elastic region of the stress-strain curve), measure the strain, and divide one by the other. In reality, moduli measured as slopes of stress-strain curves are inaccurate, often by a factor of 2 or more, because of contributions to the strain from material creep or deflection of the test machine. Accurate moduli are measured dynamically: by exciting the natural vibrations of a beam or wire, or by measuring the velocity of longitudinal or shear sound waves in the material.

Drilling down: the origins of moduli. Atoms bond together, some weakly, some strongly. If they bind strongly enough they form solids; the stronger the bond, the higher is the melting point of the solid. Think of the bonds as little springs (Figure 3). The atoms have an equilibrium spacing a_0 ; a force F pulls them apart a little, to $a_0 + \delta$, but when it is released they jump back to their original spacing. The same happens in compression because the energy of the bond increases no matter in which direction the force is applied, as the lower part of the figure suggests. The bond energy is a minimum at the equilibrium spacing. A spring that stretches by δ under a force F has a stiffness, S , defined by

$$S = \frac{F}{\delta}$$

and this is the same in compression as in tension.

Table 1 lists the stiffnesses of the different bond types; these stiffnesses largely determine the value of the modulus, E . The covalent bond is particularly stiff ($S = 20\text{--}200\text{ N/m}$); diamond, for instance, has a very high modulus because the carbon atom is small (giving a high bond density) and its atoms are linked by the stiffest springs ($S = 200\text{ N/m}$). The metallic bond is a little less stiff ($S = 15\text{--}100\text{ N/m}$) and metal atoms are often close-packed, giving metals high moduli too, though not as high as that of diamond. Ionic bonds, found in many ceramics, have stiffnesses comparable with those of metals, giving them, also, high moduli. Polymers contain both strong diamond-like covalent bonds along the polymer chain and weak hydrogen or Van-der-Waals bonds ($S = 0.5\text{--}2\text{ N/m}$) between the chains; it is the weak bonds that stretch when the polymer is deformed, giving them low moduli.

When a force F is applied to a pair of atoms, they stretch apart by δ . A force F applied to an atom corresponds to a stress $\sigma = F/a_0^2$ where a_0 is the atom spacing. A stretch δ between two

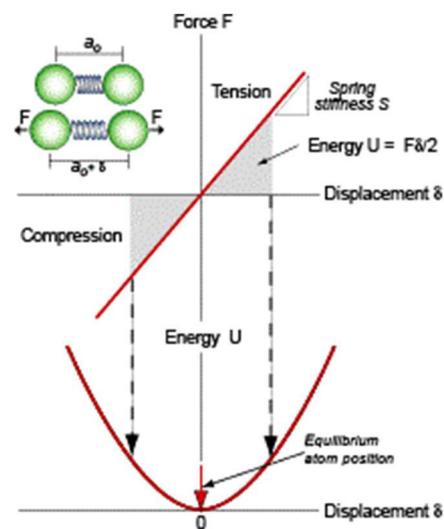


Figure 3. Stretching or compressing an atomic bond raises its energy. Its resistance to stretch is its stiffness, S .

atoms separated by a_0 corresponds to a strain $\epsilon = \delta / a_0$. Substituting these into the last equation gives

$$\sigma = \frac{S}{a_0} \epsilon$$

Table 1 Bond stiffnesses, S

Bond type	Examples	Bond Stiffness S (N/m)	Young's Modulus E (GPa)
Covalent	Carbon-carbon bond	50 – 180	200 – 1000
Metallic	All metals	15 – 75	60 – 300
Ionic	Alumina, Al ₂ O ₃	8 – 24	32 – 96
Hydrogen bond	Polyethylene	6 – 3	2 – 12
Van der Waals	Waxes	0.5 - 1	1 - 4

Comparing this with the definition of Young's modulus reveals that E is roughly

$$E = \frac{S}{a_0}$$

The largest atoms ($a_0 = 4 \times 10^{-10}$ m) bonded with the weakest bonds ($S = 0.5$ N/m) will have a modulus of roughly

$$E = \frac{0.5}{4 \times 10^{-10}} \approx 1 \text{ GPa}$$

This is the *lower limit* for true solids and many polymers do have moduli of about this value; metals and ceramics have values 50–1000 times larger because, as Table 1 shows, their bonds are stiffer.

One class of materials – elastomers (rubber) – have moduli that are much less than 1 GPa. An elastomer is a tangle of long-chain molecules with occasional cross-links, as in Figure 4 (a), as explained in **Density and atom packing**. The bonds between the molecules, apart from the cross-links, are weak – so weak that, at room temperature, they have melted. We describe this by saying that the glass temperature T_g of the elastomer – the temperature at which the bonds first start to melt – is below room temperature. Segments are free to slide over each other, and were it not for the cross-links, the material would have no stiffness at all.

Temperature favors randomness. That is why crystals melt into disordered fluids at their melting point. The tangle of Figure 4 (a) has high randomness, or expressed in the terms of thermodynamics, its *entropy* is high. Stretching it, as at (b), aligns the molecules – some parts of it now begin to resemble the crystallites shown in the notes on **Density and atom packing**. Crystals are ordered, the opposite of randomness; their entropy is low. The effect of temperature is to try to restore disorder, making the material try to revert to a random tangle, and the cross-links give it a “memory” of the disordered shape it had to start with. So there is a resistance to stretching – a stiffness – that has nothing to do with bond-stretching, but with strain-induced ordering. A full theory is complicated – it involves the statistical mechanics of long-chain tangles – so it is not easy to calculate the value of the modulus. The main thing to know is that the moduli of elastomers are low because they have this strange origin and that they increase with temperature (because of the increasing tendency to randomness), whereas those of true solids decrease (because of thermal expansion).

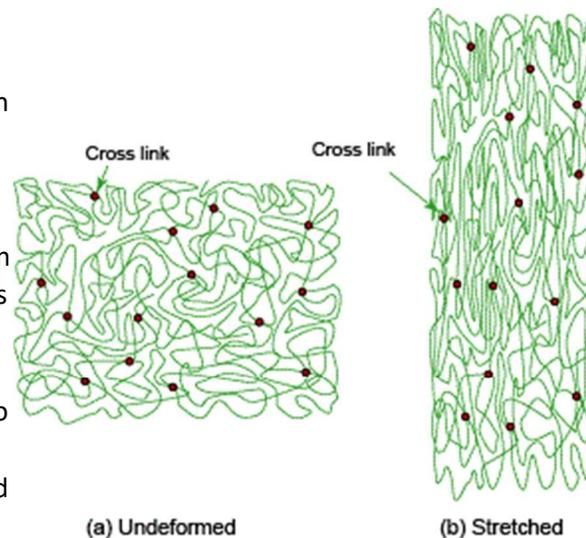


Figure 4. The stretching of an elastomer causes alignment, producing crystal-like regions. Thermal vibration drives the structure back to the one on the left, restoring its shape.

Further reading.

Author	Title	Chapter
Ashby et al	Materials: Engineering, Science, Processing and Design	4, 5
Ashby & Jones	Engineering Materials Vol 1 & 2	Vol. 1, Chap. 3, 6, 7
Askeland & Wright	The Science and Engineering of Materials	6
Budinski	Engineering Materials: Properties and Selection	2
Callister & Rethwisch	Materials Science and Engineering: An Introduction	6
Callister & Rethwisch	Fundamentals of Materials Science and Engineering: An Integrated Approach	7
Callister & Rethwisch	Materials Science and Engineering	8
Shackelford	Introduction to Materials Science for Engineers	6